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Heavy Traffic Analysis for a Multiplexer
Driven by $M|GI|\infty$ Input Processes

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Heavy traffic analysis for a multiplexer driven by $M|GI|\infty$ input processes

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Abstract

We study the heavy traffic regime of a multiplexer driven by correlated inputs, namely the $M|GI|\infty$ input processes of Cox. We distinguish between $M|GI|\infty$ processes exhibiting short or long–range dependence, identifying for each case the appropriate heavy traffic scaling that results in non-degenerate limits. As expected, the limits we obtain for short–range dependent inputs involve the standard Brownian motion. Of particular interest though are our conclusions for the long–range dependent case: The normalized queue length can be expressed as a function not of a fractional Brownian motion, but of some other stable non–Gaussian self–similar process. Thus, the $M|GI|\infty$ processes serve as an example demonstrating that, within long–range dependence, fractional Brownian motion does not assume the ubiquitous role that its counterpart, standard Brownian motion, plays in the short–range dependence setup, and that modeling possibilities attracted to non–Gaussian limits are not so hard to come by.

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1 Introduction

The apparent presence of long–range dependence and self–similarity in network traffic, that has been established in numerous studies of WAN [21], Ethernet [13] and VBR video [3] measurements, raises the need to revisit various performance analysis and design issues in this new regime. As some recent experimental work ([7]) already suggests, long–range dependence has a tangible and adverse effect on queueing measures such as buffer overflow probabilities, and consequently its presence is not to be overlooked or underestimated.

Roughly speaking, this phenomenon of long–range dependence amounts to correlations in the packet stream spanning over multiple time scales, which, although individually rather small, decay so slowly that they are non–summable in the long run, thus affecting performance in a drastically different manner than that predicted by traditional, summable correlations, Markovian models. In an effort to gain some understanding into long–range dependence, and in particular into the fundamentals of its impact on queueing, various traffic models that account for non–summable correlation patterns have been proposed. These models include, among others, fractional Gaussian noise inputs [1], fractional Brownian motion [16] and on/off sources with Pareto interrenewal periods [5], with early results yielding buffer asymptotics of radically different nature than the exponential tails, that typically characterize Markovian models.

In this paper, we consider the class of $M|GI|\infty$ input processes, which fall in the category of models that can account both for short and long–range dependent behaviour. An $M|GI|\infty$ input process is understood as the busy server process of a discrete–time infinite server system fed by a discrete–time Poisson process of rate λ (customers/slot) and with generic service time σ. Such $M|GI|\infty$ processes have already been used by Paxson and Floyd to successfully model WAN traffic in [21]. What is more, a very attractive justification for $M|GI|\infty$ modeling is given by the fact that, as it is argued in [14] and [20], these processes emerge simply by combining traffic generated by on/off sources with a Pareto distributed activity period, and letting the number of sources go to infinity, so as to obtain a behaviour identical to that of a $M|GI|\infty$ input stream with Pareto distributed σ.

The $M|GI|\infty$ processes are very flexible and allow for direct control of their correlation patterns through σ (see Proposition 2.1 and (2.1)). They are also extremely tractable; a variety of buffer tail probabilities for a multiplexer fed by $M|GI|\infty$ inputs are obtained in [18] and [20]. Surprisingly enough though, in the long–range

dependence setup, these results provide hyperbolic buffer asymptotics, altogether different from the Weibullian tails obtained under the fractional Brownian motion assumption. Here, we will try to further explore this fact, this time by developing some queueing theory for an $M|GI|\infty$ process feeding a multiplexer that operates in heavy traffic. Our motivation for going to heavy traffic is twofold : First, given that under short–range dependence heavy traffic analysis has offered useful standard Brownian motion characterizations of queueing networks, it would be interesting to try to extend the theory to the long–range dependence setup. This would possibly furnish evidence demonstrating that fractional Brownian motion, the long–range dependent analog of standard Brownian motion, plays a similar central role in the modeling of long–range dependent traffic. Secondly, for the particular case of $M|GI|\infty$ inputs, a heavy traffic analysis would help elucidate the observed differences in buffer asymptotics between $M|GI|\infty$ and fractional Gaussian noise inputs. Unless these differences are due to some fundamental structural discrepancies, one would expect that, in the heavy traffic regime, both models would collapse to a fractional Brownian motion characterization, similarly to what happens under short–range dependence, where different models eventually collapse to a description involving the standard Brownian motion. On the other hand, if the abovementioned differences are due to some different salient features of the $M|GI|\infty$ and fractional Gaussian noise process, it would be reasonable to expect that, despite the asymptotically identical correlation patterns, the differences would carry over and manifest themselves even more clearly in the heavy traffic regime.

The results of our investigation confirm the latter statement. First, we find that, under short–range dependence, the class of $M|GI|\infty$ inputs belongs to the domain of attraction of the standard Brownian motion, as expected. Next, and most importantly, we show that under long–range dependence, in particular for a σ in the domain of attraction of a non–normal stable law, the $M|GI|\infty$ process is *not* attracted to a fractional Brownian motion. Instead, the normalized heavy traffic queue length is expressed as a function of a non–Gaussian α–stable self–similar process. This result underlines the fundamentally different nature of the long-range dependent $M|GI|\infty$ processes and also points to the fact that fractional Brownian motion does not necessarily play the same key role that standard Brownian motion assumes under short–range dependence. Within long–range dependence, there seems to be a choice for distinct modeling possibilities; and, as our study suggests, it is not at all difficult to encounter a rather simple, potentially useful model, that is

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attracted to non–Gaussian limits.

The rest of the paper is organized as follows: The class of $M|GI|\infty$ input processes, along with some of their properties, is introduced in Section 2. We present our multiplexer model in the heavy traffic regime in Section 3. Our heavy traffic results are then stated in Section 4, while Section 5 addresses their implications. Finally, we give an outline of our proofs in Section 6.

A few words about the notation used here. All RVs are defined on some probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, with $\mathbf{E}$ denoting the corresponding expectation operator. Weak convergence is denoted by $\Longrightarrow$. We write $f \sim g$ when $\lim_{x \to \infty} f(x)/g(x) = 1$, and $f_r \asymp g_r$ when $\lim_{r \to \infty} f_r(x) = \lim_{r \to \infty} g(x)$, $x \in \mathbb{R}$.

2 $M|GI|\infty$ processes

As we intend to model short and long–range dependence through the busy server process $\{b_n, \ n = 0, 1, \ldots\}$ of a discrete–time $M|GI|\infty$ system, we collect in this section some pertinent facts about $\{b_n, \ n = 0, 1, \ldots\}$, that have already been established elsewhere ([6], [18], [19]), and will be used in the sequel.

To describe the process $\{b_n, \ n = 0, 1, \ldots\}$, consider a system with infinitely many servers : During time slot $[n, n + 1)$, β_{n+1} new customers arrive into the system. Customer j, $j = 1, \ldots, \beta_{n+1}$, is presented to its own server and begins service by the start of slot $[n + 1, n + 2)$; its service time has duration $\sigma_{n+1,j}$. Then b_n denotes the number of busy servers, or equivalently of customers still present in the system, at the beginning of slot $[n, n + 1)$, with b denoting the number of busy servers initially present in the system at $n = 0$.

The $I\!N$–valued RVs b, $\{\beta_{n+1}, \ n = 0, 1, \ldots\}$ and $\{\sigma_{n,j}, \ n = 0, 1, \ldots; \ j = 0, 1, \ldots\}$ satisfy the following assumptions: (i) The RVs are mutually independent; (ii) The RVs $\{\beta_{n+1}, \ n = 0, 1, \ldots\}$ are *i.i.d.* Poisson RVs with parameter $\lambda > 0$; (iii) The RVs $\{\sigma_{n,j}, \ n = 1, \ldots; \ j = 1, 2, \ldots\}$ are *i.i.d.* with common pmf G on $\{1, 2, \ldots\}$. We denote by σ a generic $I\!N$–valued RV distributed according to the pmf G. Throughout we shall assume that $\mathbf{E}[\sigma] < \infty$.

It is also helpful to introduce the forward recurrence RV $\hat{\sigma}$ associated with σ. This follows the equilibrium distribution of σ, i.e.,

$$\Gamma_n \equiv \mathbf{P}[\hat{\sigma} > n] = \frac{1}{\mathbf{E}[\sigma]} \sum_{j=n}^{\infty} \mathbf{P}[\sigma > j], \quad n = 0, 1, \ldots \qquad (2.1)$$

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and, as part 2 of the following proposition shows, it directly controls the correlations in the $\{b_n, \ n = 0, 1, \ldots\}$ stream.

Proposition 2.1 *The busy server process $\{b_n, \ n = 0, 1, \ldots\}$ admits a stationary ergodic version, still denoted by $\{b_n, \ n = 0, 1, \ldots\}$, with the following properties:*

1. *For each $n = 0, 1, \ldots$, the RV b_n is a Poisson RV with parameter $\lambda \mathbf{E}[\sigma]$.*

2. *Its covariance function is*

$$\text{cov}(b_n, b_j) = \lambda \mathbf{E}[\sigma] \Gamma_{|n-j|}, \quad n, j = 0, 1, \ldots$$

where $\Gamma_n, \ n = 0, 1, \ldots$, is given by (2.1) .

Additional information about the stationary busy server process $\{b_n, \ n = 0, 1, \ldots\}$ is contained in the characteristic function of the partial sums of the process, i.e., in

$$\Phi_r(\theta) \equiv \mathbf{E}\left[\exp\left(i\theta \sum_{n=1}^{r} b_n\right)\right] \quad r = 1, 2, \ldots, \quad \theta \in \mathbb{R}$$

Since this quantity plays instrumental role in our results, we will list below the complete expression, whose derivation can be found in [18] and [19] :

Proposition 2.2 *For each $r = 1, 2, \ldots$, and $\theta \in \mathbb{R}$, it holds that*

$$\begin{aligned}
\ln \Phi_r(\theta) &= -\lambda \mathbf{E}[\sigma]\left(1 - \mathbf{E}\left[e^{i\theta \min(r, \widehat{\sigma}-1)}\right]\right) + \lambda \mathbf{E}\left[(r - \sigma)^+ e^{i\theta\sigma}\right] \\
&\quad -\lambda r + \lambda\left(1 - e^{-i\theta}\right)^{-1} \mathbf{E}\left[e^{i\theta \min(r,\sigma)} - 1\right]
\end{aligned} \tag{2.2}$$

Having presented the necessary facts about the $M|GI|\infty$ busy server process, we next describe our heavy traffic setup.

3 The multiplexer in heavy traffic

We view the multiplexer as a discrete–time single server queue with infinite buffer capacity, operating at a constant rate and in a first–come first–served manner: Let Q_n denote the number of cells remaining in the buffer by the end of slot $[n - 1, n)$, and let b_{n+1} denote the number of new cells which arrive at the start of time slot $[n, n + 1)$. If the multiplexer output link can transmit c cells/slot, then the buffer content sequence $\{Q_n, \ n = 0, 1, \ldots\}$ evolves according to the Lindley recursion

$$Q_0 = 0; \quad Q_{n+1} = [Q_n + b_{n+1} - c]^+, \quad n = 0, 1, \ldots \tag{3.1}$$

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The stationary busy server process of a discrete–time $M|GI|\infty$ system we discussed in section 2, will now be used to model the arrival process $\{b_n, \ n = 1, 2, \ldots, \ldots\}$, thus providing us with a very simple way of accounting for time dependencies in the cell input stream. So, with the $M|GI|\infty$ process $\{b_n, \ n = 1, 2, \ldots, \ldots\}$ being characterized by the pair (λ, σ), the average input rate to the multiplexer is $\mathbf{E}[b_n] = \lambda \mathbf{E}[\sigma]$, and it can be shown that the system is stable if $\lambda \mathbf{E}[\sigma] < c$.

Our goal here is to examine the behaviour of a system that is almost fully utilized. This typically involves obtaining limiting expressions of properly rescaled quantities of interest, as the traffic intensity ρ tends towards its critical value 1. Although, in general, it may be possible to derive different flavors of heavy traffic results, one rather standard technique for studying the behaviour of a queue in heavy traffic is to consider, instead of a single queueing system, a family of queueing systems, indexed by an integer parameter $r = 1, 2, \ldots$. By taking, for the rth system, the input and the service processes to suitably depend on the parameter r, it is possible to ensure that, as $r \to \infty$, the corresponding traffic intensity ρ_r goes to 1. Adapting this approach to our context, we consider, instead of just (3.1), a family of Lindley recursions where, for each $r = 1, 2, \ldots$

$$Q_0^r = 0; \quad Q_{n+1}^r = [Q_n^r + b_{n+1}^r - c]^+, \quad n = 0, 1, \ldots \qquad (3.2)$$

Note that the stationary $M|GI|\infty$ busy server processes $\{b_n^r, n = 1, 2, \cdots\}$, $r = 1, 2, \ldots$, above are described by the Poisson arrival rate and service RV pair (λ_r, σ). That is, in our parametrization, every member of this family of queueing systems depends on r only through the input process, and in particular its Poisson rate parameter λ_r, while the service RV σ and the multiplexer release rate c remain the same.

It can be shown that the Lindley recursions (3.2) admit the following equivalent representation, which is useful for establishing heavy traffic limit theorems: By setting, for each $r = 1, 2, \ldots$,

$$S_0^r \equiv 0; \quad S_n^r \equiv \sum_{j=1}^{n} b_j^r - nc, \quad n = 1, 2, \ldots$$

we can write

$$Q_n^r =_{st} S_n^r - \inf\{S_j^r, \ j = 0, 1, \ldots, n\}, \quad n = 1, 2, \ldots$$

Before putting ourselves in the heavy traffic regime by letting the traffic intensity go to 1, it is necessary to rescale in a meaningful manner, both in time and in state

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space, the processes of interest. We do so by introducing the scaling ζ_r, $r = 1, 2, \ldots,$ to define the continuous time processes

$$q_t^r \equiv \frac{Q_{[tr]}^r}{\zeta_r}, \quad t \geq 0$$

and

$$s_t^r \equiv \frac{S_{[tr]}^r}{\zeta_r}, \quad t \geq 0$$

so that

$$q_t^r = s_t^r - \inf_{0 \leq x \leq t}\{s_x^r\}, \quad t \geq 0 \tag{3.3}$$

for every $r = 1, 2, \ldots$. The stationary version can be obtained by letting $t \to \infty$ and then $r \to \infty$, in order to drive the system to heavy traffic. Here, we will interchange the order of the limits, taking $r \to \infty$ first and $t \to \infty$ afterwards, to again collect the same stationary RV (see [11]). For a classical $GI|GI|1$ queueing setup, where second moments are always assumed to be finite, the scaling that gives rises to non-degenerate limits is $\zeta_r = \sqrt{r}$. In our $M|GI|\infty$ situation though, that allows for an infinite second moment of σ, the question arises as to what should the suitable scaling be when the input process exhibits long–range dependence. Although the answer is not immediately obvious, it is easy to see that, in all cases, to avoid collecting only a law of large numbers result, any candidate ζ_r should obey the following necessary condition :

Condition 3.1 *The sequence ζ_r, $r = 1, 2, \ldots$, satisfies*

$$\lim_{r \to \infty} \zeta_r = +\infty \quad \text{and} \quad \lim_{r \to \infty} \frac{\zeta_r}{r} = 0.$$

Without any further requirements on ζ_r, we may now enforce the heavy traffic assumption below, which guarantees that, as $r \to \infty$, the family of queueing systems described by (3.2) gradually approaches instability.

Assumption 3.1 *The sequence λ_r, $r = 1, 2, \ldots$, satisfies*

$$\lim_{r \to \infty} (\lambda_r \mathbf{E}\,[\sigma] - c)\,\frac{r}{\zeta_r} = -\gamma$$

for some $\gamma > 0$.

4 The heavy traffic results

In the heavy traffic setup of the previous section, our first task is to identify the appropriate scalings ζ_r, $r = 1, 2, \ldots$, for the various choices of the distribution of the RV σ, that controls the correlations in the input cell stream. Clearly, our aim is to obtain, as $r \to \infty$, a non-trivial limiting process $\{s_t^\infty, t \geq 0\}$. Fully characterizing this process, especially in the somewhat intriguing case of long–range dependence, will then provide information about the buffer content distribution through (3.3).

With Assumption 3.1 being enforced throughout, we distinguish between three different cases, which correspond to different tail behaviour of the distribution of σ. It is worth pointing out here that, for each case, the scaling ζ_r is essentially unique, in the sense that any other scaling ζ_r' that yields a non-degenerate limit is related to ζ_r according to $\zeta_r' = O(\zeta_r)$.

Our first result addresses the case where the $M|GI|\infty$ process is short–range dependent, i.e., $\sum_{n=1}^\infty \Gamma_n < +\infty$, or, equivalenty (see [18]), $\mathbf{E}\left[\sigma^2\right] < +\infty$.

Proposition 4.1 *(Short–range dependence) If* $\mathbf{E}\left[\sigma^2\right] < +\infty$ *then with* $\zeta_r = \sqrt{r}$, $r = 1, 2, \ldots$, *it holds that*

$$\{s_t^r, t \geq 0\} \Longrightarrow_r \left\{-\gamma t + \sqrt{c(1 + 2 \sum_{n=1}^\infty \Gamma_n)} \, B_t, \ t \geq 0\right\}$$

where $\{B_t, t \geq 0\}$ *is a standard Brownian motion.*

The next result deals with what turns out to be a cross-over case between short and long–range dependence. Despite the fact that $\mathbf{E}\left[\sigma^2\right] = +\infty$ and, as a consequence, the sum of the correlations is $\sum_{n=1}^\infty \Gamma_n = +\infty$, σ is still attracted to a Gaussian RV, while $\sum_{n=1}^r \Gamma_n$ is a slowly varying function.

Proposition 4.2 *Let* $\mathbf{E}\left[\sigma^2\right] = +\infty$ *and* $\mathbf{E}\left[\mathbf{1}\left[\sigma < x\right]\sigma^2\right] \sim 2\mathbf{E}\left[\sigma\right]h(x)$, *where* $h(x)$ *is a slowly varying function. Then with* $\zeta_r = \sqrt{rh(r)}$, $r = 1, 2, \ldots$, *it holds that*

$$\{s_t^r, t \geq 0\} \Longrightarrow_r \left\{-\gamma t + \sqrt{c} \, B_t, \ t \geq 0\right\}$$

where $\{B_t, t \geq 0\}$ *is a standard Brownian motion.*

We finally come to the bona fide long–range dependent case, where the infinite sum of correlations $\sum_{n=1}^\infty \Gamma_n = +\infty$ is induced by a σ that has a tail behaving like $h(x)x^{-\alpha}$, $1 < \alpha < 2$, where $h(x)$ is a slowly varying function. As a result, σ belongs to the domain of attraction of a non–Gaussian stable law ([8], [10], [22]) with index α, a fact denoted by $\sigma \in \mathcal{D}(\alpha)$.

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Proposition 4.3 *(Long–range dependence) Assume that $\sigma \in \mathcal{D}(\alpha)$, $1 < \alpha < 2$, so that $\mathbf{P}\left[\sigma > x\right] \sim h(x)x^{-\alpha}$, where $h(x)$ is slowly varying. By taking ζ_r to be the $1/\alpha$–regularly varying function satisfying*

$$\lim_{r \to \infty} \frac{r}{\zeta_r^\alpha} h(\zeta_r) = (\alpha - 1)\mathbf{E}\left[\sigma\right]$$

it holds that, $\forall\; t \geq 0$,

$$s_t^r \Longrightarrow_r -\gamma t + X_t$$

where X_t is a stable $S_\alpha\left(\left(ct\Gamma(2 - \alpha)\cos(\pi\frac{2-\alpha}{2})\right)^{\frac{1}{\alpha}}, 1, 0\right)$ RV.

We sketch the proofs of Propositions 4.1, 4.2 and 4.3 in Section 6, where, under long–range dependence, we take on a representative Pareto–α case. A short discussion, however, of Proposition 4.3 is in order here. It is well known (see e.g. [12], [2]) that the convergence of a normalized partial sum process, such as $\{s_t^r, t \geq 0\}$, can only be to a self–similar process, and that the Hurst parameter H may be determined through the regularly varying scaling ζ_r by

$$\lim_{r \to \infty} \frac{\zeta_{rx}}{\zeta_r} = x^H, \quad x > 0$$

Recall that, as it is mentioned in [14] and [18], with a Pareto–α σ the $M|GI|\infty$ process already possesses the so-called 2nd order asymptotic self–similarity property, with parameter $(3 - \alpha)/2$. That is, by aggregating the original process $\{b_n, n = 0, 1, \ldots\}$ in blocks of size m and dividing by the block size, one can obtain, in the limit as m goes to infinity, the same correlation function as that of a fractional Brownian motion. Because of this fact it would be tempting to think that, except perhaps for a slowly varying factor, the appropriate scaling squared is given by the rate of growth of the partial sums variance, $r^{3-\alpha}$, and that convergence occurs to a fractional Brownian motion. Proposition 4.3 however shows that this is not true and that the asymptotic 2nd order self–similarity property is misleading: The proper scaling does not exhibit any $r^{(3-\alpha)/2}$ dependence; instead it contains the $r^{1/\alpha}$ factor associated with the limit theorem for the RV σ, when $\sigma \in \mathcal{D}(\alpha)$, where $1 < \alpha < 2$. The marginal distribution of the limiting process is not Gaussian, but stable with index α, $1 < \alpha < 2$. So, the limiting process $\{s_t^\infty, t \geq 0\}$ is not a fractional Brownian motion but an α–stable self–similar process with infinite variance, Hurst parameter $H = 1/\alpha$ and dependent increments, i.e., it is not the α–stable Lévy motion, for which the increments are independent. Further work on determining the dependence structure of this process is currently in progress.

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5 Implications

We dealt with a model of a multiplexer fed by $M|GI|\infty$ processes, that account for strong dependencies in the input cell stream. Our study contributes to the analysis of such non-standard queueing systems, with long–range dependence, in the heavy traffic regime, where traditional queueing has provided useful diffusion approximations. Related work is also reported in [5], [16] and [23]. Note that in the work of Norros ([16]) the presence of fractional Brownian motion is postulated, while in [5] Brichet et al. show how fractional Brownian motion can arise naturally in the heavy traffic analysis of on/off sources with long–range dependence. Given that the $M|GI|\infty$ processes also arise as the superposition of infinitely many on/off sources (see [14]), it behoves one to ask whether our results, that belie the presence of fractional Brownian motion, contradict those of [5]. This apparent contradiction is resolved though, by observing that, apart from the different way in which heavy traffic is achieved in [5], the link to fractional Brownian motion is established for a particular choice of the on/off periods, yielding a subset of on/off sources that is distinctly different from that considered in [14], which leads to the $M|GI|\infty$ processes. Thus, our results stress that, in contrast with short–range dependence, within long–range dependence itself there exist modeling possibilities that are of fundamentally different nature. Some are attracted to fractional Brownian motion, giving rise to Weibullian buffer asymptotics, yet others are not; we have shown that the $M|GI|\infty$ model is attracted to a α–stable self–similar process, where the buffer asymptotics are expected to decay like $x^{1-\alpha}$. And in that sense, it seems that the question as to what is the effect of long–range dependence does not have a clear-cut answer, since this may depend heavily on the particular choice of model.

6 Proofs

We provide here an outline of the calculations that determine the marginal distribution of the limiting process $\{s_t^\infty, t \geq 0\}$. First, under Assumption 3.1, it is easy to conclude that

$$\lim_{r \to \infty} \mathbf{E}\left[s_t^r\right] = -\gamma t, \quad 0 \leq t \leq T$$

To alleviate notation we will look at $t = 1$; any other t can also be handled similarly. To that end, let us consider the characteristic function

$$\psi_r(\theta) \equiv \mathbf{E}\left[\exp\left(i\theta s_1^r\right)\right], \quad \theta \in \mathbb{R}, \quad r = 1, 2, \ldots \tag{6.1}$$

and the auxiliary quantity

$$\Lambda_r(\theta) \equiv \ln \mathbf{E}\left[\exp\left(\frac{i\theta}{\zeta_r} \sum_{n=1}^{r} (b_n^r - \lambda_r \mathbf{E}[\sigma]) \right) \right], \quad \theta \in \mathbb{R}, \quad r = 1, 2, \ldots \quad (6.2)$$

By writing

$$s_1^r = \frac{1}{\zeta_r} \sum_{n=1}^{r} (b_n^r - \lambda_r \mathbf{E}[\sigma]) + \frac{r}{\zeta_r}(\lambda_r \mathbf{E}[\sigma] - c), \quad r = 1, 2, \ldots \quad (6.3)$$

it follows that

$$\psi_r(\theta) = \exp\left(\Lambda_r(\theta) + i\theta \frac{r}{\zeta_r}(\lambda_r \mathbf{E}[\sigma] - c) \right), \quad r = 1, 2, \ldots \quad (6.4)$$

In order to avoid a degenerate limiting RV, we are interested in selecting ζ_r, $r = 1, 2, \ldots$, such that the limit

$$\Lambda(\theta) \equiv \lim_{r \to \infty} \Lambda_r(\theta), \quad \theta \in \mathbb{R} \quad (6.5)$$

exists, is non-zero and finite, in which case, under Assumption 3.1, so does the limiting characteristic function

$$\psi(\theta) \equiv \lim_{r \to \infty} \psi_r(\theta), \quad \theta \in \mathbb{R} \quad (6.6)$$

So, provided that the limit (6.5) exists and Assumption 3.1 holds, by using (6.4), $\psi(\theta)$ can be expressed as

$$\psi(\theta) = e^{-i\theta\gamma + \Lambda(\theta)}, \quad \theta \in \mathbb{R} \quad (6.7)$$

After some algebra, Proposition 2.2 enables us to write

$$
\begin{aligned}
\Lambda_r(\theta) &= -\lambda_r \mathbf{E}[\sigma] r \frac{i\theta}{\zeta_r} + \lambda_r \mathbf{E}[\sigma] r \left(\exp\left(\frac{i\theta}{\zeta_r} \right) - 1 \right) \\
&\quad \cdot \left\{ 1 + \left(1 - \exp\left(-\frac{i\theta}{\zeta_r} \right) \right) \sum_{n=1}^{r} \left(1 - \frac{n}{r} \right) \Gamma_n \exp\left(\frac{i\theta}{\zeta_r} n \right) \right\} \\
&= -\lambda_r \mathbf{E}[\sigma] r \frac{i\theta}{\zeta_r} + \lambda_r \mathbf{E}[\sigma] i\theta \, \frac{\exp\left(\frac{i\theta}{\zeta_r} \right) - 1}{\frac{i\theta}{\zeta_r}} \frac{r}{\zeta_r} \\
&\quad \cdot \left\{ 1 + \frac{1 - \exp\left(-\frac{i\theta}{\zeta_r} \right)}{\frac{i\theta}{\zeta_r}} \frac{i\theta}{\zeta_r} \frac{1}{r} \sum_{n=1}^{r} (r - n)\Gamma_n \exp\left(\frac{i\theta}{\zeta_r} n \right) \right\} \\
&= \lambda_r \mathbf{E}[\sigma] i\theta \, \frac{r}{\zeta_r} \left(\frac{\exp\left(\frac{i\theta}{\zeta_r} \right) - 1}{\frac{i\theta}{\zeta_r}} - 1 \right) - \lambda_r \mathbf{E}[\sigma] \theta^2 \, \frac{\exp\left(\frac{i\theta}{\zeta_r} \right) - 1}{\frac{i\theta}{\zeta_r}} \\
&\quad \cdot \frac{1 - \exp\left(-\frac{i\theta}{\zeta_r} \right)}{\frac{i\theta}{\zeta_r}} \frac{1}{\zeta_r^2} \sum_{n=1}^{r} (r - n)\Gamma_n \exp\left(\frac{i\theta}{\zeta_r} n \right) \quad (6.8)
\end{aligned}
$$

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To ease the computations, we introduce the quantities

$$K_r(\theta) \equiv r \left(1 + \frac{i\theta}{\zeta_r} - \exp\left(\frac{i\theta}{\zeta_r}\right) \right), \quad \theta \in \mathbb{R}, \quad r = 1, 2, \ldots \tag{6.9}$$

and

$$F_r(\theta) \equiv \theta^2 \frac{1}{\zeta_r^2} \sum_{n=1}^{r} (r - n)\Gamma_n \exp\left(\frac{i\theta}{\zeta_r}n\right), \quad \theta \in \mathbb{R}, \quad r = 1, 2, \ldots \tag{6.10}$$

In all cases, we will seek ζ_r, $r = 1, 2, \ldots$, ensuring that the limits

$$K(\theta) \equiv \lim_{r \to \infty} K_r(\theta), \quad \theta \in \mathbb{R} \tag{6.11}$$

and

$$F(\theta) \equiv \lim_{r \to \infty} F_r(\theta), \quad \theta \in \mathbb{R} \tag{6.12}$$

exist and are finite. Since, by Condition 3.1, $\lim_{r \to \infty} \zeta_r = +\infty$, it follows that

$$\lim_{r \to \infty} \frac{\exp\left(\frac{i\theta}{\zeta_r}\right) - 1}{\frac{i\theta}{\zeta_r}} = \lim_{r \to \infty} \frac{1 - \exp\left(-\frac{i\theta}{\zeta_r}\right)}{\frac{i\theta}{\zeta_r}} = 1 \tag{6.13}$$

and, by virtue of Assumption 3.1 and (6.9) – (6.12), the limit (6.5) of (6.8) can be rewritten more compactly as

$$\Lambda(\theta) = -c\left(K(\theta) + F(\theta)\right), \quad \theta \in \mathbb{R} \tag{6.14}$$

As we have already mentioned, the choice of ζ_r, $r = 1, 2, \ldots$, should yield $K(\theta)$ and $F(\theta)$ such that $F(\theta) + K(\theta)$ is not identically zero.

Short–range dependence ($\sum_{n=1}^{\infty} \Gamma_n < +\infty$)

With the scaling $\zeta_r = \sqrt{r}$, $r = 1, 2, \ldots$, (6.9) and (6.11) yield

$$K(\theta) = \lim_{r \to \infty} r \left(\frac{\theta^2}{2r} - o\left(\frac{\theta^2}{r}\right) \right) = \frac{\theta^2}{2}, \quad \theta \in \mathbb{R} \tag{6.15}$$

To compute $F(\theta)$, recall first that, since $\Gamma_n \geq 0$, $n = 1, 2, \ldots$, the implication

$$\sum_{n=1}^{\infty} \Gamma_n < +\infty \quad \Rightarrow \quad \lim_{n \to \infty} n\Gamma_n = 0 \tag{6.16}$$

holds true and, by Cesaro convergence, we then have

$$\lim_{r \to \infty} \frac{1}{r} \left| \sum_{n=1}^{r} n\Gamma_n \exp\left(\frac{i\theta}{\sqrt{r}}n\right) \right| \leq \lim_{r \to \infty} \frac{1}{r} \sum_{n=1}^{r} n\Gamma_n = 0, \quad \theta \in \mathbb{R} \tag{6.17}$$

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Applying this fact to the limit (6.12), where in (6.10) we have inserted the scaling $\zeta_r = \sqrt{r}$, gives

$$F(\theta) = \theta^2 \lim_{r \to \infty} \sum_{n=1}^{r} \Gamma_n \exp\left(\frac{i\theta}{\sqrt{r}}n\right), \quad \theta \in \mathbb{R} \tag{6.18}$$

In order to show that the exponential factor can effectively be replaced by 1 in $F(\theta)$ above, we form the difference below and bound it, for $r_\varepsilon = 1, 2, \ldots$, by

$$\left| \sum_{n=1}^{r} \Gamma_n \left(\exp\left(\frac{i\theta}{\sqrt{r}}n\right) - 1 \right) \right|$$

$$\leq \sum_{n=1}^{r_\varepsilon} \Gamma_n \left| \exp\left(\frac{i\theta}{\sqrt{r}}n\right) - 1 \right| + \sum_{n=r_\varepsilon}^{r} \Gamma_n \left| \exp\left(\frac{i\theta}{\sqrt{r}}n\right) - 1 \right|$$

$$\leq \sum_{n=1}^{r_\varepsilon} \left| \exp\left(\frac{i\theta}{\sqrt{r}}n\right) - 1 \right| + 2 \sum_{n=r_\varepsilon}^{r} \Gamma_n \tag{6.19}$$

Now, since for fixed r_ε it holds that

$$\lim_{r \to \infty} \sum_{n=1}^{r_\varepsilon} \left| \exp\left(\frac{i\theta}{\sqrt{r}}n\right) - 1 \right| = 0, \quad \theta \in \mathbb{R}$$

by invoking the fact that

$$\sum_{n=1}^{\infty} \Gamma_n < +\infty \;\Rightarrow\; \sum_{n=r_\varepsilon}^{\infty} \Gamma_n < \varepsilon, \quad \forall\, \varepsilon > 0$$

for r_ε large enough, we see that each of the two terms in (6.19) can be made arbitrarily small, so as to yield

$$\lim_{r \to \infty} \left| \sum_{n=1}^{r} \Gamma_n \left(\exp\left(\frac{i\theta}{\sqrt{r}}n\right) - 1 \right) \right| = 0, \quad \theta \in \mathbb{R}$$

therefore $F(\theta)$ of (6.18) is simply

$$F(\theta) = \theta^2 \sum_{n=1}^{\infty} \Gamma_n, \quad \theta \in \mathbb{R} \tag{6.20}$$

Substituting (6.20) and (6.15) in (6.14) and inserting this in (6.7) we obtain a Gaussian characteristic function

$$\psi(\theta) = \exp\left(-i\theta\gamma - \frac{1}{2}\theta^2 c(1 + 2\sum_{n=1}^{\infty} \Gamma_n) \right), \quad \cdot\theta \in \mathbb{R} \tag{6.21}$$

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Long–range dependence $\left(\sum_{n=1}^{\infty} \Gamma_n = +\infty\right)$

We illustrate the results under long–range dependence by focusing on a σ that follows a Pareto-like distribution with parameter α, where $1 < \alpha \leq 2$. In particular, we take

$$\mathbf{P}\left[\sigma > n\right] = \frac{(n+1)^{1-\alpha} - (n+2)^{1-\alpha}}{1 - 2^{1-\alpha}}, \quad n = 0, 1, \dots$$

for which

$$\mathbf{E}\left[\sigma\right] = \frac{1}{1 - 2^{1-\alpha}}$$

and

$$\lim_{n \to \infty} \frac{\mathbf{P}\left[\sigma > n\right]}{n^{-\alpha}} = (\alpha - 1)\mathbf{E}\left[\sigma\right]$$

Consequently, the distribution of the forward recurrence time $\hat{\sigma}$ associated with σ is given by

$$\Gamma_n \equiv \mathbf{P}\left[\hat{\sigma} > n\right] = \frac{1}{\mathbf{E}\left[\sigma\right]} \sum_{j=n}^{\infty} \mathbf{P}\left[\sigma > j\right] = (n+1)^{1-\alpha}, \quad n = 0, 1, \dots$$

We begin by noting that the case $\alpha = 2$ needs to be treated separately (Proposition 4.2), because even though $\mathbf{E}\left[\sigma^2\right]$ is infinite, σ still belongs to the domain of attraction of the normal distribution (but not in the domain of the so-called normal attraction of the normal distribution, for which $\mathbf{E}\left[\sigma^2\right] < +\infty$) . The input process $\{b_n^r, n = 1, 2, \cdots\}$ exhibits long–range dependence, yet is barely does so, since $\sum_{n=1}^{r} \Gamma_n \sim \ln r \equiv h(r)$ is slowly varying. So, undertaking the $\alpha = 2$ case first, we proceed to show that the appropriate heavy traffic scaling is $\zeta_r = \sqrt{rh(r)} = \sqrt{r \ln r}$, $r = 1, 2, \dots$. We begin by substituting this scaling in (6.9), so that (6.11) reads

$$
\begin{aligned}
K(\theta) &= \lim_{r \to \infty} r \left(1 + \frac{i\theta}{\sqrt{r \ln r}} - \exp\left(\frac{i\theta}{\sqrt{r \ln r}}\right)\right) = \lim_{r \to \infty} r \left(\frac{\theta^2}{2r \ln r} - o\left(\frac{\theta^2}{r \ln r}\right)\right) \\
&= \lim_{r \to \infty} \frac{1}{\ln r} \left(\frac{\theta^2}{2} - \frac{o\left(\frac{\theta^2}{r \ln r}\right)}{\frac{1}{r \ln r}}\right) = 0, \quad \theta \in \mathbb{R}
\end{aligned}
\tag{6.22}
$$

Next, consider (6.10), which in this case becomes

$$
\begin{aligned}
F_r(\theta) &= \theta^2 \frac{1}{r \ln r} \sum_{n=1}^{r-1} \frac{r-n}{n+1} \exp\left(\frac{i\theta n}{\sqrt{r \ln r}}\right) \\
&= \theta^2 \frac{1}{\ln r} \sum_{n=1}^{r-1} \frac{1}{n+1} \exp\left(\frac{i\theta n}{\sqrt{r \ln r}}\right) - \theta^2 \frac{1}{r \ln r} \sum_{n=1}^{r-1} \frac{n}{n+1} \exp\left(\frac{i\theta n}{\sqrt{r \ln r}}\right) \\
&= \theta^2 \frac{1}{\ln r} \sum_{n=1}^{r} \frac{1}{n} \exp\left(\frac{i\theta n}{\sqrt{r \ln r}}\right) \exp\left(-\frac{i\theta}{\sqrt{r \ln r}}\right)
\end{aligned}
$$

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$$-\theta^2 \frac{1}{\ln r} - \theta^2 \frac{1}{r \ln r} \sum_{n=1}^{r-1} \frac{n}{n+1} \exp\left(\frac{i\theta n}{\sqrt{r \ln r}}\right)$$

for $r = 2, 3, \ldots$, $\theta \in \mathbb{R}$. Since

$$\left| \sum_{n=1}^{r-1} \frac{n}{n+1} \exp\left(\frac{i\theta n}{\sqrt{r \ln r}}\right) \right| < r - 1, \quad \theta \in \mathbb{R}$$

it is clear that only the first sum will yield a non-zero contribution to the limit, so we have

$$
\begin{aligned}
F_r(\theta) &\asymp \theta^2 \frac{1}{\ln r} \sum_{n=1}^{r} \frac{1}{n} \exp\left(\frac{i\theta n}{\sqrt{r \ln r}}\right) \\
&= \theta^2 \frac{1}{\ln r} \left(\sum_{n=1}^{r} \frac{1}{n} + i \int_0^{\theta/\sqrt{r \ln r}} \sum_{n=1}^{r} e^{inx} \, dx \right) \\
&\asymp \theta^2 \left(1 + i \frac{1}{\ln r} \int_0^{\theta/\sqrt{r \ln r}} e^{ix} \frac{e^{irx} - 1}{e^{ix} - 1} \, dx \right) \\
&= \theta^2 \left(1 + i \frac{1}{\ln r} \int_0^{\theta/\sqrt{r \ln r}} \frac{e^{irx} - 1}{1 - e^{-ix}} \, dx \right), \quad \theta \in \mathbb{R}
\end{aligned}
\tag{6.23}
$$

One can verify that the denominator in the integral above can be rewritten as $1/(1 - e^{-ix}) = \frac{1}{2}(1 - i \cot(x/2))$ and then it is easy to see that, because $|e^{irx} - 1| \le 2$, we get

$$\lim_{r \to \infty} \left| \int_0^{\theta/\sqrt{r \ln r}} \left(e^{irx} - 1 \right) \, dx \right| \le \lim_{r \to \infty} \frac{2|\theta|}{\sqrt{r \ln r}} = 0, \quad \theta \in \mathbb{R}$$

so that $F_r(\theta)$ of (6.23) is

$$F_r(\theta) \asymp \theta^2 \left(1 + \frac{1}{2 \ln r} \int_0^{\theta/\sqrt{r \ln r}} \left(e^{irx} - 1 \right) \cot(x/2) \, dx \right), \quad \theta \in \mathbb{R} \tag{6.24}$$

In Lemma A.1 we essentially show that (6.24) asymptotically remains unchanged if we replace $\frac{1}{2} \cot(x/2)$ by $\frac{1}{x}$, thus leading us to the expression

$$
\begin{aligned}
F(\theta) &= \theta^2 \left(1 + \lim_{r \to \infty} \frac{1}{\ln r} \int_0^{\theta/\sqrt{r \ln r}} \frac{e^{irx} - 1}{x} \, dx \right) \\
&= \theta^2 \left(1 + \lim_{r \to \infty} \frac{1}{\ln r} \int_0^{\theta\sqrt{\frac{r}{\ln r}}} \frac{e^{ix} - 1}{x} \, dx \right), \quad \theta \in \mathbb{R}
\end{aligned}
\tag{6.25}
$$

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The evaluation of the limit is supplied in Lemma A.2. Inserting the result in (6.25), combining with (6.22) in (6.14) and finally in (6.7) yields once more a Gaussian characteristic function :

$$\psi(\theta) = \exp\left(-i\theta\gamma - \frac{1}{2}\theta^2 c\right), \quad \theta \in \mathbb{R} \tag{6.26}$$

We subsequently study the case where $1 < \alpha < 2$ (Proposition 4.3), noting that σ is now in the domain of attraction of a non–normal stable law. Of primary importance in this long–range dependence setup is the parameter H, which we define as

$$H \equiv \frac{1}{\alpha}, \quad 1 < \alpha < 2 \tag{6.27}$$

We will show that the appropriate scaling turns out to be $\zeta_r = r^H$, $r = 1, 2, \ldots$, giving rise to a non–Gaussian, α–stable, self–similar process $\{s_t^\infty, t \geq 0\}$, whose Hurst parameter H is given by (6.27). Observe that $1 < \alpha < 2$ will yield H in the range $1/2 < H < 1$, while in our previous calculations, where $\alpha = 2$, the emerging self–similar process was a standard Brownian motion, whose corresponding H is exactly $1/2$.

Starting from $K(\theta)$, we have

$$
\begin{aligned}
K(\theta) &= \lim_{r\to\infty} r\left(1 + \frac{i\theta}{r^H} - \exp\left(\frac{i\theta}{r^H}\right)\right) = \lim_{r\to\infty} r\left(\frac{\theta^2}{2r^{2H}} - o\left(\frac{\theta^2}{r^{2H}}\right)\right) \\
&= \lim_{r\to\infty} r^{1-2H}\left(\frac{\theta^2}{2} - \frac{o\left(\frac{\theta^2}{r^{2H}}\right)}{\frac{1}{r^{2H}}}\right) = 0, \quad \theta \in \mathbb{R} \tag{6.28}
\end{aligned}
$$

Turning to $F_r(\theta)$, $r = 2, 3, \ldots$, we write

$$
\begin{aligned}
F_r(\theta) &= \theta^2 \frac{1}{r^{2H}} \sum_{n=1}^{r-1}(r-n)(n+1)^{1-\alpha}\exp\left(\frac{i\theta}{r^H}n\right) \\
&= \theta^2 \frac{1}{r^{2H}} \sum_{n=2}^{r}(r-n+1)n^{1-\alpha}\exp\left(\frac{i\theta}{r^H}(n-1)\right) \\
&= \theta^2 \frac{1}{r^{2H}} \sum_{n=1}^{r}(rn^{1-\alpha} - n^{2-\alpha})\exp\left(\frac{i\theta}{r^H}(n-1)\right) \\
&\quad + \theta^2 \frac{1}{r^{2H-1}} \frac{1}{r} \sum_{n=1}^{r} n^{1-\alpha}\exp\left(\frac{i\theta}{r^H}(n-1)\right) - \theta^2 \frac{1}{r^{2H-1}}
\end{aligned}
$$

Since $1 < \alpha < 2$, we have $0 < 2H - 1 < 1$, therefore the third term above vanishes as $r \to \infty$ and, by Cesaro convergence, so does the second. We can then rearrange

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the only contributing summand in the limit (6.12) as

$$F(\theta) = \lim_{r \to \infty} r^{H(\alpha-1)(2-\alpha)} \sum_{n=1}^{r} \left(|\theta|^\alpha \left(\frac{n|\theta|}{r} \right)^{1-\alpha} \frac{|\theta|}{r} \right.$$
$$\left. -|\theta|^{\alpha-1} \left(\frac{n|\theta|}{r} \right)^{2-\alpha} \frac{|\theta|}{r} \right) \exp \left(ir^{1-H} \operatorname{sgn}(\theta) \frac{n|\theta|}{r} \right), \quad \theta \in \mathbb{R}$$

Clearly, since $F(\theta)$ and $F(-\theta)$ are complex conjugates, it suffices to carry out the calculations only for $\theta \geq 0$. Introducing the staircase function $T_r : \mathbb{R} \mapsto \mathbb{R}_+$

$$T_r(x) \equiv \begin{cases} \frac{n}{r} \theta & \frac{n-1}{r} \theta < x \leq \frac{n}{r} \theta, \quad \theta \geq 0 \quad n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

and letting $k_r \equiv r^{1-H}$, $r = 1, 2, \dots$, it is not difficult to check that $F(\theta)$, $\theta \geq 0$ assumes the form

$$F(\theta) = \theta^\alpha \lim_{r \to \infty} r^{H(\alpha-1)(2-\alpha)} \int_0^\theta T_r(x)^{1-\alpha} \exp\left(ik_r \theta T_r(x)\right) \, dx \tag{6.29}$$
$$-\theta^{\alpha-1} \lim_{r \to \infty} r^{H(\alpha-1)(2-\alpha)} \int_0^\theta T_r(x)^{2-\alpha} \exp\left(ik_r \theta T_r(x)\right) \, dx, \quad \theta \geq 0$$

Application of Lemma A.3 to the each of the two limits in (6.29) implies that we can replace $T_r(x)$ by x, and, as a result, $F(\theta)$ becomes

$$F(\theta) = \lim_{r \to \infty} r^{H(\alpha-1)(2-\alpha)} \int_0^\theta \left(\theta^\alpha x^{1-\alpha} - \theta^{\alpha-1} x^{2-\alpha} \right) \exp\left(ik_r \theta x\right) \, dx, \quad \theta \geq 0$$

By defining the quantities

$$I_{1-\alpha}(\theta) \equiv \theta^\alpha \lim_{r \to \infty} r^{H(\alpha-1)(2-\alpha)} \int_0^\theta x^{1-\alpha} \exp\left(ik_r \theta x\right) \, dx, \quad \theta \geq 0 \tag{6.30}$$

and

$$I_{2-\alpha}(\theta) \equiv \theta^{\alpha-1} \lim_{r \to \infty} r^{H(\alpha-1)(2-\alpha)} \int_0^\theta x^{2-\alpha} \exp\left(ik_r \theta x\right) \, dx, \quad \theta \geq 0 \tag{6.31}$$

we can express $F(\theta)$ as

$$F(\theta) = I_{1-\alpha}(\theta) - I_{2-\alpha}(\theta), \quad \theta \geq 0 \tag{6.32}$$

Appealing to the results of Lemma A.4, where we carry out the detailed calculation for $I_{1-\alpha}(\theta)$ and $I_{2-\alpha}(\theta)$, and recalling that $F(\theta)$ and $F(-\theta)$ are complex conjugates, we finally obtain

$$F(\theta) = |\theta|^\alpha \Gamma(2 - \alpha) \exp\left(i\pi \operatorname{sgn}(\theta) \frac{2-\alpha}{2} \right), \quad \theta \in \mathbb{R} \tag{6.33}$$

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Upon inserting (6.33) and (6.28) in (6.14), then this in (6.7) and rearranging, we collect the limiting characteristic function

$$\psi(\theta) = \exp\left(-i\theta\gamma - c|\theta|^{\alpha}\Gamma(2-\alpha)\cos(\pi\frac{2-\alpha}{2})\left(1 - i\operatorname{sgn}(\theta)\tan(\frac{\pi\alpha}{2})\right)\right), \quad \theta \in \mathbb{R}$$

(6.34)

which corresponds to a stable $S_{\alpha}\left(\left(c\Gamma(2-\alpha)\cos(\pi\frac{2-\alpha}{2})\right)^{\frac{1}{\alpha}}, 1, -\gamma\right)$ RV (see [22]).

A Appendix

Lemma A.1 *If* $\lim_{r\to\infty} \delta_r = 0$ *then*

$$\lim_{r\to\infty} \int_0^{\delta_r} \frac{e^{irx} - 1}{x}\left(\frac{x}{2}\cot\frac{x}{2} - 1\right) dx = 0$$

(A.1)

Proof. The Taylor expansion of the function $x \mapsto x\cot x$ gives $x\cot x = 1 - x^2/3 + o(x^2)$, which enables us to write

$$\frac{1}{x}\left(\frac{x}{2}\cot\frac{x}{2} - 1\right) = \frac{x}{12} + \frac{o(x^2)}{x} = x\left(\frac{1}{12} + \frac{o(x^2)}{x^2}\right)$$

Now, recall that $\forall\, \varepsilon > 0\ \exists\, x_{\varepsilon} > 0$ such that for $|x| < x_{\varepsilon}$ it holds that $o(x^2)/x^2 < \varepsilon$ and, by $\lim_{r\to\infty} \delta_r = 0$, $\exists\, r_{\varepsilon}$ such that when $r > r_{\varepsilon}$ we have $|\delta_r| < x_{\varepsilon}$. As a result, we conclude that

$$\left|\frac{1}{x}\left(\frac{x}{2}\cot\frac{x}{2} - 1\right)\right| \le \left(\frac{1}{12} + \varepsilon\right)|\delta_r|, \quad r > r_{\varepsilon}, \quad |x| < \delta_r$$

(A.2)

Combining (A.2) with $|e^{irx} - 1| \le 2$ and with our assumption $\lim_{r\to\infty} \delta_r = 0$, quickly leads us to (A.1). ∎

Lemma A.2 *It holds that*

$$\lim_{r\to\infty} \frac{1}{\ln r} \int_0^{\theta\sqrt{\frac{r}{\ln r}}} \frac{e^{ix} - 1}{x}\, dx = -\frac{1}{2}, \quad \theta \ne 0$$

(A.3)

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Proof. The integrals for θ and $-\theta$ are complex conjugates, so it suffices to establish (A.3) for $\theta > 0$. Let us define $R \equiv \theta\sqrt{\frac{r}{\ln r}}$ and write, for large r,

$$
\int_0^R \frac{e^{ix}-1}{x}\,dx = \int_0^1 \frac{e^{ix}-1}{x}\,dx + \int_1^R \frac{e^{ix}-1}{x}\,dx
$$

$$
= \int_0^1 \frac{e^{ix}-1}{x}\,dx - \ln R + \int_1^R \frac{e^{ix}}{x}\,dx \qquad (A.4)
$$

It follows that only the $\ln R$ term above will offer any contribution to the limit in (A.3). To see this note that the first term is easily bounded as

$$
\left| \int_0^1 \frac{e^{ix}-1}{x}\,dx \right| \leq 1 \qquad (A.5)
$$

while the last one can be handled by a contour integration :

$$
0 = \int_{-R}^{-1} \frac{e^z}{z}\,dz + \int_{e^{i\pi}}^{e^{i\pi/2}} \frac{e^z}{z}\,dz + \int_i^{iR} \frac{e^z}{z}\,dz + \int_{Re^{i\pi/2}}^{Re^{i\pi}} \frac{e^z}{z}\,dz
$$

$$
= \int_{-R}^{-1} \frac{e^x}{x}\,dx + i\int_\pi^{\pi/2} \exp\left(\cos\phi + i\sin\phi\right)\,d\phi + \int_1^R \frac{e^{ix}}{x}\,dx
$$

$$
+i\int_{\pi/2}^{\pi} \exp\left(R(\cos\phi + i\sin\phi)\right)\,d\phi
$$

This implies that

$$
\left| \int_1^R \frac{e^{ix}}{x}\,dx \right| \leq \int_1^R \frac{e^{-x}}{x}\,dx + \int_{\pi/2}^{\pi} e^{\cos\phi}\,d\phi + \int_{\pi/2}^{\pi} e^{R\cos\phi}\,d\phi \qquad (A.6)
$$

where all the terms in the right hand side are bounded uniformly in R. Then, because of (A.5) and (A.6), (A.4) will yield the desired conclusion (A.3), after multiplying by $1/\ln r$, taking the limit $r \to +\infty$ and using the fact that

$$
\lim_{r\to\infty} \frac{\ln\left(\theta\sqrt{\frac{r}{\ln r}}\right)}{\ln r} = -\frac{1}{2}, \quad \theta > 0
$$

■

Lemma A.3 *For $1 < \alpha < 2$, $p > -1$, $H \equiv \frac{1}{\alpha}$ and $\theta \geq 0$ it holds that*

$$
\lim_{r\to\infty} r^{H(\alpha-1)(2-\alpha)} \int_0^\theta \left(T_r(x)^p \exp\left(ik_r\theta T_r(x)\right) - x^p \exp\left(ik_r\theta x\right)\right)\,dx = 0 \qquad (A.7)
$$

Proof. We first bound the difference by

$$\left| \int_0^\theta T_r(x)^p \exp\left(ik_r\theta T_r(x)\right) \, dx - \int_0^\theta x^p \exp\left(ik_r\theta x\right) \, dx \right|$$

$$\leq \left| \int_0^\theta x^p \left(\exp\left(ik_r\theta T_r(x)\right) - \exp\left(ik_r\theta x\right)\right) \, dx \right|$$

$$+ \left| \int_0^\theta \left(T_r(x)^p - x^p\right) \exp\left(ik_r\theta T_r(x)\right) \, dx \right| \qquad (A.8)$$

and then bound each of the two terms separately. Recalling that $k_r \equiv r^{1-H}$ and $T_r(x) - x < \frac{\theta}{r}$, $r = 1, 2, \ldots$, the first term is bounded as follows :

$$\left| \int_0^\theta x^p \left(\exp\left(ik_r\theta T_r(x)\right) - \exp\left(ik_r\theta x\right)\right) \, dx \right|$$

$$\leq \int_0^\theta x^p \left|\exp\left(ik_r\theta(T_r(x) - x)\right) - 1\right| \, dx$$

$$\leq \int_0^\theta x^p \sum_{n=1}^\infty \frac{k_r^n}{n!} \left|T_r(x) - x\right|^n \, dx$$

$$\leq \int_0^\theta x^p \sum_{n=1}^\infty \frac{k_r^n}{n!} \frac{\theta^n}{r^n} \, dx$$

$$= \left(e^{\frac{k_r}{r}\theta} - 1\right) \int_0^\theta x^p \, dx$$

$$= \frac{\theta^{p+1}}{p+1} \left(e^{\theta r^{-H}} - 1\right) \qquad (A.9)$$

We multiply by $r^{H(\alpha-1)(2-\alpha)}$ and calculate the limit :

$$\lim_{r\to\infty} r^{H(\alpha-1)(2-\alpha)} \left(e^{\theta r^{-H}} - 1\right) = \lim_{r\to\infty} r^{H(\alpha-1)(2-\alpha)} \left(\theta r^{-H} + o\left(r^{-H}\right)\right)$$

$$= \lim_{r\to\infty} \frac{r^{H(\alpha-1)(2-\alpha)}}{r^H} \left(\theta + \frac{o\left(r^{-H}\right)}{r^{-H}}\right)$$

$$= \lim_{r\to\infty} r^{-H\left(2-\alpha+(\alpha-1)^2\right)} \left(\theta + \frac{o\left(r^{-H}\right)}{r^{-H}}\right)$$

$$= 0 \qquad (A.10)$$

For the second term in (A.8), with $p > -1$, we have

$$\left| \int_0^\theta \left(T_r(x)^p - x^p\right) \exp\left(ik_r\theta T_r(x)\right) \, dx \right| \leq \int_0^\theta \left(T_r(x)^p - x^p\right) \, dx$$

$$\leq \sum_{n=1}^{r} \left(\frac{n\theta}{r}\right)^p \frac{\theta}{r} - \frac{\theta^{p+1}}{p+1}$$

$$\leq \frac{\theta^{p+1}}{r^{p+1}} \int_0^{r+1} x^p \, dx - \frac{\theta^{p+1}}{p+1}$$

$$= \frac{\theta^{p+1}}{r^{p+1}} \frac{(r+1)^{p+1}}{p+1} - \frac{\theta^{p+1}}{p+1}$$

$$= \frac{\theta^{p+1}}{p+1} \left(\left(1+\frac{1}{r}\right)^{p+1} - 1\right)$$

$$(A.11)$$

Again, multiplying by $r^{H(\alpha-1)(2-\alpha)}$ and writing $(1+\frac{1}{r})^{p+1} - 1 = \frac{p+1}{r} + o\left(\frac{1}{r}\right)$ we calculate the limit as

$$\lim_{r\to\infty} r^{H(\alpha-1)(2-\alpha)} \left(\left(1+\frac{1}{r}\right)^{p+1} - 1\right) = \lim_{r\to\infty} r^{H(\alpha-1)(2-\alpha)} \left(\frac{p+1}{r} + o\left(\frac{1}{r}\right)\right)$$

$$= \lim_{r\to\infty} \frac{r^{H(\alpha-1)(2-\alpha)}}{r} \left(p+1 + \frac{o\left(\frac{1}{r}\right)}{\frac{1}{r}}\right)$$

$$= 0 \qquad\qquad (A.12)$$

where the last equality is derived easily by observing that, whenever $1 < \alpha < 2$, $r^{H(\alpha-1)(2-\alpha)} = o(r)$. After multiplying (A.8) by $r^{H(\alpha-1)(2-\alpha)}$ and taking the limit as $r \to +\infty$ we invoke (A.12), (A.11) and (A.10), (A.9) to obtain (A.7). ∎

Lemma A.4 *For $1 < \alpha < 2$ it holds that*

$$I_{1-\alpha}(\theta) = \theta^\alpha \Gamma(2-\alpha) \exp\left(i\pi\theta\frac{2-\alpha}{2}\right), \quad \theta \geq 0 \qquad (A.13)$$

and

$$I_{2-\alpha}(\theta) = 0, \quad \theta \geq 0 \qquad (A.14)$$

Proof. Both statements are clearly true when $\theta = 0$. To prove (A.13) for $\theta > 0$ we start from definition (6.30) and perform a change of variable $u = k_r x$ to get

$$I_{1-\alpha}(\theta) = \theta^\alpha \lim_{r\to\infty} \int_0^{\theta k_r} u^{1-\alpha} e^{iu\theta} \, du, \quad \theta > 0 \qquad (A.15)$$

Writing $R \equiv \theta k_r$, $r = 1, 2, \ldots$, for the upper limit and observing that $\lim_{r \to \infty} R = +\infty$, $\theta > 0$, this integral can be evaluated by the standard first quadrant contour integration below:

$$
\begin{aligned}
0 &= \int_{r_0}^{R} z^{1-\alpha} e^{iz} \, dz + \int_{Re^{i0}}^{Re^{i\pi/2}} z^{1-\alpha} e^{iz} \, dz + \int_{iR}^{ir_0} z^{1-\alpha} e^{iz} \, dz + \int_{r_0 e^{i\pi/2}}^{r_0 e^{i0}} z^{1-\alpha} e^{iz} \, dz \\
&= \int_{r_0}^{R} x^{1-\alpha} e^{ix} \, dx + iR^{2-\alpha} \int_{0}^{\pi/2} \exp\left(i\phi(2 - \alpha) + iRe^{i\phi} \right) \, d\phi \\
&\quad + i^{2-\alpha} \int_{R}^{r_0} y^{1-\alpha} e^{-y} \, dy + ir_0^{2-\alpha} \int_{\pi/2}^{0} \exp\left(i\phi(2 - \alpha) + ir_0 e^{i\phi} \right) \, d\phi
\end{aligned}
$$

Letting r_0 go to 0, we have

$$
\begin{aligned}
\int_{0}^{R} x^{1-\alpha} e^{ix} \, dx &= -iR^{2-\alpha} \int_{0}^{\pi/2} \exp\left(i\phi(2 - \alpha) + iRe^{i\phi} \right) \, d\phi \\
&\quad + i^{2-\alpha} \int_{0}^{R} y^{1-\alpha} e^{-y} \, dy
\end{aligned}
\tag{A.16}
$$

Observe that the last integral above yields a Gamma function :

$$
i^{2-\alpha} \lim_{r \to \infty} \int_{0}^{R} y^{1-\alpha} e^{-y} \, dy = -e^{-i\frac{\pi\alpha}{2}} \Gamma(2 - \alpha), \quad \theta > 0
\tag{A.17}
$$

Next, calculate the limit involving the other remaining integral in (A.16), using the bound

$$
\left| \int_{0}^{\pi/2} \exp\left(i\phi(2 - \alpha) + iRe^{i\phi} \right) \, d\phi \right| \leq \int_{0}^{\pi/2} \exp\left(-R \sin \phi \right) \, d\phi
\tag{A.18}
$$

and the fact that $\sin \phi \geq \frac{2\phi}{\pi}$, $0 \leq \phi \leq \pi/2$, which is implied by the concavity of $\phi \mapsto \sin \phi - \frac{2\phi}{\pi}$, for $\phi \in [0, \frac{\pi}{2}]$:

$$
\begin{aligned}
\lim_{r \to \infty} R^{2-\alpha} \int_{0}^{\pi/2} \exp\left(-R \sin \phi \right) \, d\phi &\leq \lim_{r \to \infty} R^{2-\alpha} \int_{0}^{\pi/2} \exp\left(-\frac{2R}{\pi} \phi \right) \, d\phi \\
&= \frac{\pi}{2} \lim_{r \to \infty} R^{2-\alpha} \frac{1 - e^{-R}}{R} \\
&= \frac{\pi}{2\theta} \lim_{r \to \infty} R^{1-\alpha} \left(1 - e^{-R} \right) \\
&= 0, \quad \theta > 0
\end{aligned}
\tag{A.19}
$$

Taking the limit as $r \to \infty$ of (A.16), combining (A.17), (A.18), (A.19) and inserting in (A.15) readily yields (A.13) for $\theta > 0$.

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Having shown (A.13), it is easy to prove the claim for $I_{2-\alpha}(\theta)$ by integrating by parts :

$$\int_0^\theta x^{2-\alpha} e^{ik_r x}\, dx = \frac{1}{ik_r}\left(\theta^{2-\alpha} e^{ik_r \theta} - (2-\alpha)\int_0^\theta x^{1-\alpha} e^{ik_r x}\, dx\right) \qquad (A.20)$$

After multiplying the equation above by $r^{H(\alpha-1)(2-\alpha)}$, (A.14) follows from

$$
\begin{aligned}
|I_{2-\alpha}(\theta)| &\leq \theta^{\alpha-1}\left(\lim_{r\to\infty}\frac{r^{H(\alpha-1)(2-\alpha)}}{k_r}\,\theta^{2-\alpha} + \lim_{r\to\infty}\frac{1}{k_r}(2-\alpha)\theta^{-\alpha}|I_{1-\alpha}(\theta)|\right) \\
&= \theta\lim_{r\to\infty} r^{-\frac{(\alpha-1)^2}{2}} + \frac{1}{\theta}(2-\alpha)|I_{1-\alpha}(\theta)|\lim_{r\to\infty}\frac{1}{k_r} \\
&= 0, \qquad \theta > 0
\end{aligned}
$$

∎

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